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Quantum Nonthermal Radiation of the Nonstationary Kerr–Newman Black Hole

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Quantum nonthermal radiation of a nonstationary Kerr–Newman black hole is studied by using the Hamilton–Jacobi equation and generalized tortoise coordinates. It is shown that the positive energy state interlaces with the negative state in a region near the event horizon, and spontaneous quantum nonthermal radiation takes place in the overlap region.

1. INTRODUCTION

Since Hawking's original discovery of black hole thermal radiation by using techniques of quantum field theory on a given classical background spacetime,[(1)] quantum thermal radiation by black holes has been studied extensively in different types of spacetimes.[(2–8)] Recently, Jing and Wang[(9)] investigated the Hawking radiation of a nonstationary Kerr–Newman black hole.

In addition to quantum thermal radiation, there is also important quantum nonthermal radiation in the spacetimes of the some black holes. In present paper we study the nonthermal radiation of a nonstationary Kerr–Newman black hole. The investigation of the nonthermal radiation of the black hole is interesting because it yields some new properties and includes the results of some well-known black holes.

2. SPACETIME OF A NONSTATIONARY KERR–NEWMAN BLACK HOLE

The line element of the nonstationary Kerr–Newman spacetime represented in advanced Eddington coordinates is as follows[(10)]:

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$$
\begin{aligned}
ds^2 = {} & [1 - (2Mr - Q^2)\rho\bar{\rho}]\, dv^2 + 2\, dv\, dr \\
& + 2a(2Mr - Q^2)\rho\bar{\rho} \sin^2\theta\, dv\, d\phi - 2a \sin^2\theta\, dr\, d\phi \\
& - \frac{1}{\rho\bar{\rho}}\, d\theta^2 + \left[(Q^2 - 2Mr)a^2\rho\bar{\rho} - \frac{r^2 + a^2}{\sin^2\theta} \right] \sin^4\theta\, d\phi^2 \qquad (1)
\end{aligned}
$$

where $\rho = -1/(r - ia \cos\theta)$ and $\bar{\rho}$ is the complex conjugate of ρ. $M(v)$ and $Q(v)$ are the mass and charge of the nonstationary Kerr–Newman black hole, respectively, and they are arbitrary functions of the retarded time coordinate v. Here a is a constant just as in the Kerr–Newman case.

The contravariant of the metric is given by

$$
g^{\mu\nu} = \begin{pmatrix} -a^2 \sin^2\theta\, \rho\bar{\rho} & (r^2 + a^2)\rho\bar{\rho} & 0 & -a\rho\bar{\rho} \\ (r^2 + a^2)\rho\bar{\rho} & (2Mr - Q^2 - r^2 - a^2)\rho\bar{\rho} & 0 & a\rho\bar{\rho} \\ 0 & 0 & -\rho\bar{\rho} & 0 \\ -a\rho\bar{\rho} & a\rho\bar{\rho} & 0 & -\rho\bar{\rho}/\sin^2\theta \end{pmatrix} \qquad (2)
$$

In the following, we first find the event horizon equation for this black hole, and then study the quantum nonthermal effect in the spacetime.

The event horizon is determined by the null surface condition. From the null surface condition

$$
n_\mu n^\mu = g^{\mu\nu} \frac{\partial f}{\partial x^\mu} \cdot \frac{\partial f}{\partial x^\nu} \qquad (3)
$$

we have

$$
\begin{aligned}
& a^2 \sin^2\theta \left(\frac{\partial f}{\partial v} \right)^2 + (r^2 + a^2 + Q^2 - 2Mr)\left(\frac{\partial f}{\partial r} \right)^2 + \left(\frac{\partial f}{\partial \theta} \right)^2 \\
& - 2(r^2 + a^2) \frac{\partial f}{\partial v} \frac{\partial f}{\partial r} = 0 \qquad (4)
\end{aligned}
$$

Introducing the generalized tortoise coordinate transformation[11]

$$
\begin{aligned}
r_* &= r + \frac{1}{2k} \ln[r - r_H(v, \theta)] \\
v_* &= v - v_0 \qquad (5) \\
\theta_* &= \theta - \theta_0
\end{aligned}
$$

we have

$$\frac{\partial}{\partial r} = \frac{2k(r - r_H) + 1}{2k(r - r_H)} \frac{\partial}{\partial r_*}$$

$$\frac{\partial}{\partial v} = \frac{\partial}{\partial v_*} - \frac{\dot{r}_H}{2k(r - r_H)} \frac{\partial}{\partial r_*}$$

$$\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta_*} - \frac{r'_H}{2k(r - r_H)} \frac{\partial}{\partial r_*} \tag{6}$$

where $\dot{r}_H = \partial r_H/\partial v$, $r'_H = \partial r_H/\partial \theta$. Then equation (4) can be written as

$$\{a^2 \sin^2\theta\, \dot{r}_H^2 + (r^2 + a^2 + Q^2 - 2Mr)[2k(r - r_H) + 1]^2 + r'^2_H$$
$$+ 2(r^2 + a^2)[2k(r - r_H) + 1]\dot{r}_H\}\left(\frac{\partial f}{\partial r_*}\right)^2 - 4k(r - r_H)\Bigg\{a^2 \sin^2\theta\, \dot{r}_H \frac{\partial f}{\partial v_*}$$
$$+ r'_H \frac{\partial f}{\partial \theta_*} + (r^2 + a^2)[2k(r - r_H) + 1] \frac{\partial f}{\partial v_*}\Bigg\}\frac{\partial f}{\partial r_*}$$
$$+ 4k^2 (r - r_H)^2 \left[a^2 \sin^2\theta \left(\frac{\partial f}{\partial v_*}\right)^2 + \left(\frac{\partial f}{\partial \theta_*}\right)^2\right] = 0 \tag{7}$$

On the event horizon surface $r = r_H$, so that we can obtain from (7)

$$a^2 \sin^2\theta\, \dot{r}_H^2 + (r_H^2 + a^2 + Q^2 - 2Mr) + r'^2_H + 2(r_H + a^2)\dot{r}_H = 0 \tag{8}$$

This is the equation that the event horizon satisfies. When $\dot{r}_H = 0$ and $r'_H = 0$, we have

$$r_H^{\pm} = M \pm \sqrt{M^2 - a^2 - Q^2} \tag{9}$$

The results reduce to the well-known stationary Kerr–Newman black hole results.

3. THE QUANTUM NONTHERMAL EFFECT

The Hamilton–Jacobi equation governing the motion of a particle with charge e and mass μ in a spacetime with metric tensor $g^{\mu\nu}$ is given by[12]

$$g^{\mu\nu}\left(\frac{\partial S}{\partial X^\mu} - eA_\mu\right)\left(\frac{\partial S}{\partial X^\nu} - eA_\nu\right) - \mu^2 = 0 \tag{10}$$

where $S = S(v, r, \theta, \phi)$ is the Hamilton principal function, and A_μ is the four-potential in the spacetime.

Considering the axisymmetry of the nonstationary Kerr–Newman black hole, we can let

$$A_\mu = (A_0, 0, 0, A_3) \tag{11}$$

Substituting (2) and (11) into the Hamilton–Jacobi equation (10), we have

$$\begin{aligned}
&a^2\sin^2\theta\left(\frac{\partial S}{\partial v}\right)^2 + (r^2 + a^2 + Q^2 - 2Mr)\left(\frac{\partial S}{\partial r}\right)^2 + \left(\frac{\partial S}{\partial \theta}\right)^2 \\
&+\frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 - 2(r^2 + a^2)\frac{\partial S}{\partial v}\frac{\partial S}{\partial r} + 2a\frac{\partial S}{\partial v}\frac{\partial S}{\partial \phi} \\
&-2a\frac{\partial S}{\partial r}\frac{\partial S}{\partial \phi} - 2e(a^2A_0\sin^2\theta + aA_3)\frac{\partial S}{\partial v} + 2e[(r^2 + a^2)A_0 + aA_3]\frac{\partial S}{\partial r} \\
&-2e\left(\frac{A_3}{\sin^2\theta} + aA_0\right)\frac{\partial S}{\partial \phi} + a^2\sin^2\theta\, e^2A_0^2 + \frac{1}{\sin^2\theta}e^2A_3^2 \\
&+2ae^2A_0A_3 + \mu^2(r^2 + a^2\cos^2\theta) = 0
\end{aligned} \tag{12}$$

Introducing the generalized tortoise coordinate transformation (5) and (6), we can write equation (12) as

$$\begin{aligned}
&a^2\sin^2\theta\left[\frac{\partial S}{\partial v_*} - \frac{\dot r_H}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]^2 \\
&+ (r^2 + a^2 + Q^2 - 2Mr)\left[\frac{2k(r - r_H) + 1}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]^2 \\
&+ \left[\frac{\partial S}{\partial \theta_*} - \frac{r'_H}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2 \\
&- 2(r^2 + a^2)\left[\frac{\partial S}{\partial v_*} - \frac{\dot r_H}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\frac{2k(r - r_H) + 1}{2k(r - r_H)}\frac{\partial S}{\partial r_*} \\
&+ 2a\left[\frac{\partial S}{\partial v_*} - \frac{\dot r_H}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right]\frac{\partial S}{\partial \phi} - 2a\frac{2k(r - r_H) + 1}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\frac{\partial S}{\partial \phi} \\
&- 2e(a^2A_0\sin^2\theta + aA_3)\left[\frac{\partial S}{\partial v_*} - \frac{\dot r_H}{2k(r - r_H)}\frac{\partial S}{\partial r_*}\right] \\
&+ 2e[(r^2 + a^2)A_0 + aA_3]\frac{2k(r - r_H) + 1}{2k(r - r_H)}\frac{\partial S}{\partial r_*} - 2e\left(\frac{A_3}{\sin^2\theta} + aA_0\right)\frac{\partial S}{\partial \phi} \\
&+a^2\sin^2\theta\, e^2A_0^2 + \frac{1}{\sin^2\theta}e^2A_3^2 + 2ae^2A_0A_3 + \mu^2(r^2 + a^2\cos^2\theta) = 0
\end{aligned} \tag{13}$$

There exists a Killing vector $(\partial/\partial\phi)^a$ in the spacetime, so that

$$\frac{\partial S}{\partial \phi} = m \quad (\text{const}) \tag{14}$$

Let us define

$$\omega = -\frac{\partial S}{\partial v_*}, \qquad l = \frac{\partial S}{\partial \theta_*} \tag{15}$$

ω and l are the energy and the angular momentum of the particle, respectively. Equation (13) can be reduced to

$$D\left(\frac{\partial S}{\partial r_*}\right)^2 - 4k(r - r_H)\, E\left(\frac{\partial S}{\partial r_*}\right) + [2k(r - r_H)]^2\, R = 0 \tag{16}$$

where

$$\begin{aligned}
D &= a^2 \sin^2\theta\, \dot{r}_H^2 + (r^2 + a^2 + Q^2 - 2Mr)[2k(r - r_H) + 1]^2 + (r'_H)^2 \\
&\quad + 2(r^2 + a^2)[2k(r - r_H) + 1]\dot{r}_H \\
E &= -\{a^2 \sin^2\theta\, \dot{r}_H \omega - r'_H\, l + (r^2 + a^2)[2k(r - r_H) + 1]\omega \\
&\quad - am\dot{r}_H - a[2k(r - r_H) + 1]m + e(a^2 A_0 \sin^2\theta + aA_3)\dot{r}_H \\
&\quad + e[(r^2 + a^2)A_0 + aA_3][2k(r - r_H) + 1]\} \qquad (17) \\
R &= a^2 \sin^2\theta\, \omega^2 + l^2 + \frac{m^2}{\sin^2\theta} - 2a\omega m + 2e(a^2 A_0 \sin^2\theta + aA_3)\omega \\
&\quad - 2e\left(\frac{A_3}{\sin^2\theta} + aA_0\right)m + a^2 \sin^2\theta\, e^2 A_0^2 + \frac{1}{\sin^2\theta}\, e^2 A_3^2 \\
&\quad + 2ae^2 A_0 A_3 + \mu^2(r^2 + a^2 \cos^2\theta)
\end{aligned}$$

The solutions of equation (16) are

$$\frac{\partial S}{\partial r_*} = \frac{2k(r - r_H)}{D}\left(E \pm \sqrt{E^2 - DR}\right) \tag{18}$$

Both S and $\partial S/\partial r_*$ must are real numbers, so that

$$E^2 - DR \geq 0 \tag{19}$$

that is,

$$\begin{aligned}
&\{a^2 \sin^2\theta\, \dot{r}_H\, \omega - r'_H l + (r^2 + a^2)[2k(r - r_H) + 1]\omega - a\dot{r}_H m \\
&\quad - a[2k(r - r_H) + 1]m + e(a^2 A_0 \sin^2\theta + aA_3)\dot{r}_H
\end{aligned}$$

$$+ e[(r^2 + a^2)A_0 + aA_3][2k(r - r_H) + 1]\}^2$$
$$- D[a^2 \sin^2\theta\, \omega^2 + l^2 + \frac{m^2}{\sin^2\theta} - 2am\omega + 2e(a^2 A_0 \sin^2\theta + aA_3)\omega$$
$$- 2e\left(\frac{A_3}{\sin^2\theta} + aA_0\right)m + a^2 \sin^2\theta\, e^2 A_0^2 + \frac{1}{\sin^2\theta} e^2 A_3^2$$
$$+ 2ae^2 A_0 A_3 + \mu^2(r^2 + a^2 \cos^2\theta)] \geq 0 \tag{20}$$

This is the relation that the energy levels of Dirac particles have to satisfy in the nonstationary Kerr–Newman spacetime. Let us adopt the equality in (20). We obtain from (20)

$$\omega^{\pm} = \frac{-F_2 \pm \sqrt{F_2^2 - F_1 F_3}}{F_1} \tag{21}$$

where

$$F_1 = \{a^2 \sin^2\theta\, \dot{r}_H + (r^2 + a^2)[2k(r - r_H) + 1]\}^2 - Da^2 \sin^2\theta$$
$$F_2 = \{a^2 \sin^2\theta\, \dot{r}_H + (r^2 + a^2)[2k(r - r_H) + 1]\}\{-\dot{r}'_H l - am\dot{r}_H$$
$$- am[2k(r - r_H) + 1] + e(a^2 A_0 \sin^2\theta + aA_3)\dot{r}_H$$
$$+ e[(r^2 + a^2)A_0 + aA_3][2k(r - r_H) + 1]\}$$
$$+ D[am - e(a^2 A_0 \sin^2\theta + aA_3)]$$
$$F_3 = \{-\dot{r}'_H l - am\dot{r}_H - am[2k(r - r_H) + 1] + e(a^2 A_0 \sin^2\theta + aA_3)\dot{r}_H$$
$$+ e[(r^2 + a^2)A_0 + aA_3][2k(r - r_H) + 1]\}^2$$
$$- D\left[l^2 + \frac{m^2}{\sin^2\theta} - 2e\left(\frac{A_3}{\sin^2\theta} + aA_0\right)m + a^2 \sin^2\theta\, e^2 A_0^2\right.$$
$$\left. + \frac{1}{\sin^2\theta} e^2 A_3^2 + 2ae^2 A_0 A_3 + \mu^2(r^2 + a^2 \cos^2\theta)\right] \tag{22}$$

The distribution of the energy levels of the Dirac vacuum is given by

$$\omega \geq \omega^+ \tag{23}$$

and

$$\omega \leq \omega^- \tag{24}$$

The forbidden region of the particle energy is

$$\omega^- < \omega < \omega^+ \tag{25}$$

The width of the forbidden region is

$$\Delta\omega = \omega^+ - \omega^- = 2\,\frac{\sqrt{F_2^2 - F_1F_3}}{F_1} \tag{26}$$

4. DISCUSSION

1. When $r \to \infty$, electromagnetic effects can be neglected, and we have

$$\omega^{\pm} \to \pm\mu \tag{27}$$

The distribution of the Dirac energy levels goes to that in the Minkowski spacetime. The width of the forbidden region is $\Delta\omega = 2\mu$.

2. Now let us consider the case near the event horizon r_H. When $r \to r_H$, we have from (17)

$$\lim_{r\to r_H} D = a^2 \sin^2\theta\, \dot{r}_H^2 + (r_H^2 + a^2 + Q^2 - 2Mr_H) + r_H'^2 + 2(r_H^2 + a^2)\dot{r}_H$$

This is just the null surface condition (8), so that

$$\lim_{r\to r_H} D = 0 \tag{28}$$

From (22) and (28), we get

$$\lim_{r\to r_H} (F_2^2 - F_1F_3) = 0 \tag{29}$$

$$\begin{aligned}\omega_0 &= \lim_{r\to r_H} \omega^+ = \lim_{r\to r_H} \omega^- = -\lim_{r\to r_H} \frac{F_2}{F_1}\\ &= \frac{am + am\dot{r}_H + lr'_H - e(a^2A_0 \sin^2\theta + aA_3)\dot{r}_H - e[(r_H^2 + a^2)A_0 + aA_3]}{(r_H^2 + a^2) + a^2 \sin^2\theta\, \dot{r}_H}\end{aligned} \tag{30}$$

The width of the forbidden region vanishes at the event horizon,

$$\lim_{r\to r_H} \Delta\omega = \lim_{r\to r_H} \frac{2\sqrt{F_2^2 - F_1F_3}}{F_1} = 0 \tag{31}$$

This means that there exists a crossing of the positive and negative energy levels near the event horizon. When $\omega_0 > +\mu$, the particle can escapes to infinity from the event horizon. Namely, there is radiation from the region near the event horizon. This quantum effect is nonthermal. It is independent of the temperature of the black hole. This is the Starobinsky–Unruh process.

From (30), we find that ω_0 depends not only on the evaporation rate ($\sim \dot{r}_H$) and the event horizon shape ($\sim r'_H$) of the black hole, but also on the four-potential A_μ in the spacetime. When $\dot{r}_H = 0$ and $r'_H = 0$ we have

$$\omega_0 = \frac{am - e[(r_H^2 + a^2)A_0 + aA_3]}{r_H^2 + a^2} \tag{32}$$

This reduces to the stationary Kerr–Newman spacetime result.

In summary, we have studied nonthermal radiation of a nonstationary Kerr–Newman black hole. The exact expressions of the energy of the positive and negative states are given. The positive energy state interlaces with the negative state in a region near the event horizon, and spontaneous nonthermal radiation takes place in the overlap region.

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